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Is the strong-interaction proton-proton scattering length renormalization scale dependent in effective field theory?

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Abstract. It is shown that the strong-interaction $^1S_0$ proton-proton scattering length in very low-energy effective field theory does not depend on the renormalization scale, if the electromagnetic interaction is "switched off" consistently.

PACS. 03.65.Nk Scattering theory – 13.75.Cs Nucleon-nucleon interactions (including antinucleons, deuterons, etc.)

Over the past decade Weinberg's papers on describing nuclear forces using chiral Lagrangians [1,2] have triggered an intensive activity (see, *e.g.*, refs. [3–5] and references therein). For processes involving more than one nucleon, Weinberg suggested to apply the power counting to the effective potentials. The transition amplitudes are then obtained by solving the Lippmann-Schwinger equation (or the Schrödinger equation). This approach has been applied to various problems involving two and three nucleons.

In this work we address the dependence of the strong-interaction proton-proton scattering length on the renormalization mass parameter encountered in ref. [6]. The discussion below closely follows the paper by X. Kong and F. Ravndal [6] and the author's PhD Thesis [7]. Similar considerations have recently been presented independently in refs. [8,9].

In order to describe proton-proton scattering at very low energies, one can integrate out all particles except protons and photons. The lowest-order strong-interaction part of the effective non-relativistic Lagrangian for protons in the spin-singlet channel reads [6]

$$\mathcal{L}_0 = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi\sigma_2\psi)(\psi\sigma_2\psi)^\dagger, \tag{1}$$

where ψ is the two-component field of the proton, M the mass of the proton, C_0 a coupling constant and σ_2 a Pauli matrix. This Lagrangian corresponds to the singular potential $C_0\delta(\mathbf{r})$ which affects interactions only in the S-wave. On top of eq. (1), one also needs to include the static Coulomb repulsion between protons. The effective strength of this repulsion is $\eta(p) \equiv \eta = \alpha M/2p$, where p is the magnitude of the CM momentum of the protons and $\alpha = 1/137$ is the fine-structure constant. For small p, η is large and hence the Coulomb repulsion becomes strong. The scattering problem for both the Coulomb repulsion and the singular strong-interaction potential of eq. (1) can be solved simultaneously using the well-established formalism based upon the exact solutions of the Schrödinger equation in the Coulomb potential [10].

In a partial-wave expansion of the full scattering amplitude [11], the total phase shifts δ_ℓ can be written as $\sigma_\ell + \delta_\ell^C$, where σ_ℓ are the pure Coulombic phase shifts. For pp S-wave scattering, δ_{pp}^C is related to the corresponding (modified) strong amplitude $T_{SC}(p)$ by the standard partial-wave expression

$$p\left(\cot\delta_{pp}^C - i\right) = -\frac{4\pi}{M}\frac{e^{2i\sigma_0}}{T_{SC}(p)}. \tag{2}$$

Note that δ_{pp}^C besides pure strong-interaction effects still contains remnants of the electromagnetic interaction. It is only the Coulomb repulsion between the protons in the initial and final states that has been removed at this stage.

It is well known that $\cot\delta_{pp}^C$ in eq. (2) does not have a regular effective range expansion. Rather, one finds [12]

$$p\left[C_\eta^2\left(\cot\delta_{pp}^C - i\right) + 2\eta H(\eta)\right] = -\frac{1}{a_{pp}^C} + \frac{1}{2}r_0\,p^2 + \ldots, \tag{3}$$

where

$$C_\eta^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1} \tag{4}$$

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is the Sommerfeld factor [10], a_{pp}^C and r_0 are the S-wave Coulomb-modified scattering length and effective range, respectively. They arise after removing the part of the amplitude described by the complex function [13]

$$H(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \ln(i\eta) \tag{5}$$

representing Coulomb effects at short distances. Here, the ψ function is the logarithmic derivative of the Γ function. The imaginary part of eq. (5) cancels the term $\sim i$ in eq. (3). The real part defines the function $h(\eta) = \mathrm{Re}\psi(i\eta) - \ln\eta$ which is more suitable for a phenomenological analysis [14].

The proton-proton scattering amplitude can be calculated from the effective Lagrangian of eq. (1) (plus the Coulomb term). It takes the form [6]:

$$T_{SC}(p) = C_\eta^2 \frac{C_0\, e^{2i\sigma_0}}{1 - C_0\, J_0(p)}, \tag{6}$$

where

$$J_0(p) = M \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{2\pi\eta(k)}{e^{2\pi\eta(k)} - 1} \frac{1}{p^2 - k^2 + i\epsilon}. \tag{7}$$

When this result for the scattering amplitude is used in eqs. (2) and (3), we see that both the phase shift σ_0 and the Sommerfeld factor C_η^2 cancel out. We are thus only left with the evaluation of eq. (7) which can be done using the power divergent subtraction (PDS) scheme of ref. [15] in $d = 3 - \epsilon$ dimensions introducing a renormalization mass μ. An ultraviolet divergence shows up as an $1/\epsilon$ pole in the integral. This will be cancelled by counterterms which renormalize the coupling C_0 in eq. (6) to $C_0(\mu)$. As a result, the finite part of the dressed bubble, eq. (7), is found to be [6]

$$J_0^{\mathrm{finite}}(p) = \frac{\alpha M^2}{4\pi} \left[\ln \frac{\mu\sqrt{\pi}}{\alpha M} + 1 - \frac{3}{2} C_{\mathrm{E}} - H(\eta) \right] - \frac{\mu M}{4\pi}, \tag{8}$$

where $C_{\mathrm{E}} = 0.5772\ldots$ is Euler's constant. The last term of eq. (8) is the contribution from the special PDS pole in $d = 2$ dimensions. We now see that also the function $H(\eta)$ cancels out in eq. (3). At this order in the effective theory there is no contribution to the effective range r_0. If one defines the strong scattering length (with the Coulomb interaction switched off) as [6]

$$\frac{1}{a_{pp}} = \frac{4\pi}{M C_0(\mu)} + \mu, \tag{9}$$

then from eqs. (3) and (6) it can be expressed in terms of the measured scattering length a_{pp}^C as

$$\frac{1}{a_{pp}} = \frac{1}{a_{pp}^C} + \alpha M \left[\ln \frac{\mu\sqrt{\pi}}{\alpha M} + 1 - \frac{3}{2} C_{\mathrm{E}} \right]. \tag{10}$$

As is seen from eq. (10), a_{pp} depends on μ. It has been argued in ref. [6] that the strong scattering length a_{pp} is not a physical quantity as it cannot be measured directly and thus in general it can depend on the renormalization point μ. However, this explanation does not seem very satisfactory if a_{pp} is indeed understood as a scattering length for the strong interaction when the Coulomb interaction is switched off. It is completely true that a_{pp} is not measurable experimentally, but from a theoretical point of view it still is a physical quantity. As is clear from the analyses of ref. [6], the sum of the δ-function and the Coulomb potential is renormalizable. The δ-function potential is also renormalizable. Hence, as one could consider these two potetials themselves as independent "models" (without higher-order corrections of effective field theory), all physical quantities of both "models" should be renormalization point independent. Consequently, one *cannot* expect that the dependence of the strong scattering length on the renormalization point can be canceled by contributions of higher-order terms in the potential generated by effective field theory as suggested in ref. [6]. The origin of the μ-dependence of a_{pp} is the α-dependence of the running of $C_0(\mu)$ in eq. (9). In order to define the strong scattering length consistently as the quantity of the theory with the electromagnetic interaction being switched off, we should also put $\alpha = 0$ in the running of $C_0(\mu)$.

It is straightforward to calculate the running of the renormalized coupling constant $C_0(\mu)$ using the results of ref. [6]:

$$C_0(\mu) = \frac{C_0\,(\mu_0)}{1 - C_0\,(\mu_0) \left\{ -\frac{\alpha M^2}{4\pi} \ln \frac{\mu}{\mu_0} + \frac{M}{4\pi}\,(\mu - \mu_0) \right\}}. \tag{11}$$

Setting $\alpha = 0$ in eq. (11) yields

$$\tilde{C}_0(\mu) = \frac{\tilde{C}_0\,(\mu_0)}{1 - \tilde{C}_0\,(\mu_0)\, \frac{M}{4\pi}\,(\mu - \mu_0)}. \tag{12}$$

Note that the value of the strong-coupling constant $\tilde{C}_0\,(\mu)$ for $\mu = \mu_0$ when the Coulomb interaction is switched off does not coincide with $C_0\,(\mu)$. Defining

$$\frac{1}{a_{pp}} = -p \left[C_\eta^2 \left(\cot\delta_{pp}^C - i \right) + 2\eta H(\eta) \right] |_{p=0,\alpha=0} = \frac{4\pi}{M\tilde{C}_0(\mu)} + \mu, \tag{13}$$

one obtains the strong scattering length which does not depend on the renormalization parameter μ. The numerical value of $\tilde{C}_0(\mu_0)$ for some fixed μ_0 has to be given as an input, as it *cannot* be calculated from $C_0\,(\mu)$ within the given effective field theory. This is in agreement with the result of ref. [16] that the $^1S_0$ pp scattering amplitude cannot be devided into strong and electromagnetic parts in a model-independent way.

Note that if we consider both protons and neutrons with an isospin invariant contact interaction in the $^1S_0$ partial wave then at a given order of accuracy $\tilde{C}_0(\mu)$ exactly coincides with the renormalized coupling of pn and nn contact interactions, and, consequently, a_{pp} coincides with $a_{pn} = a_{nn}$. This pins down $\tilde{C}_0(\mu)$. Unfortunately, this is only the manifestation of the isospin symmetry which has been taken as an input. Given $\tilde{C}_0(\mu)$ as an input (fixed through a_{nn}), one cannot calculate $C_0(\mu)$ (and, consequently, a^C_{pp}) within the given effective theory.

One can also include the next-to-leading order correction to the effective Lagrangian (1) which for the S-wave channel reads [6]

$$\frac{C_2}{16}\,(\psi\sigma_2(\overrightarrow{\nabla}-\overleftarrow{\nabla})^2\psi)(\psi\sigma_2\psi)^\dagger+\text{h.c.} \tag{14}$$

Taking into account the contribution of $\mathcal{L}_2$ to the pp potential, one obtains the amplitude (in dimensional regularization) in exact analogy to the case of the contact interactions plus one-pion exchange potential of ref. [17]:

$$T_{SC}(p)=\frac{\psi(0)^2}{\frac{1}{C_0+C_2p^2}-G_E(0,0)}, \tag{15}$$

where $\psi(0)$ is Coulomb wave function at the origin and $G_E(0,0)$ is the coordinate-space propagator from the origin to the origin in the presence of the Coulomb potential. Substituting the values of these two quantities [6] we obtain

$$T_{SC}(p)=C_\eta^2\frac{e^{2i\sigma_0}}{\frac{1}{C_0+C_2p^2}-J_0(p)}. \tag{16}$$

The model is no longer renormalizable, *i.e.* not all divergences can be absorbed in available parameters, but if one expands in powers of C_2 and keeps only the zeroth- and first-order terms (in C_2), then all divergences can be absorbed in C_0 and C_2. The renormalization is performed analogously to the case without Coulomb interaction [18,19], *i.e.* one expands

$$T_{SC}(p)=C_\eta^2\frac{e^{2i\sigma_0}}{\frac{1}{C_0}-J_0(p)}+C_\eta^2\frac{e^{2i\sigma_0}p^2\;C_2}{C_0^2\left(\frac{1}{C_0}-J_0(p)\right)^2}+O\left(p^4\;C_2^2\right) \tag{17}$$

and absorbs the divergences of $J_0(p)$ by counterterms which renormalize C_0 leading to the running coupling of eq. (11). The remaining divergences contained in the C_0^2 factor in the denominator of the second term in eq. (17) is absorbed into the renormalization of C_2 by demanding

$$\frac{C_2}{C_0^2}=\frac{C_2(\mu)}{C_0(\mu)^2}. \tag{18}$$

As the left-hand side of the eq. (18) is the ratio of bare couplings, it does not depend on μ, hence the right-hand side does not depend on μ either. Writing for some fixed μ_0,

$$\frac{C_2(\mu)}{C_0^2(\mu)}=\frac{C_2(\mu_0)}{C_0(\mu_0)^2}, \tag{19}$$

solving eq. (19) for $C_2(\mu)$ and taking eq. (11) into account, one obtains the following expression:

$$C_2(\mu)=\frac{C_2\left(\mu_0\right)}{\left(1-C_0\left(\mu_0\right)\left\{-\frac{\alpha M^2}{4\pi}\ln\frac{\mu}{\mu_0}+\frac{M}{4\pi}\left(\mu-\mu_0\right)\right\}\right)^2}. \tag{20}$$

Analogously to $C_0(\mu)$, the running of $C_2(\mu)$ depends on α. Therefore, to obtain the effective theory with electromagnetic interaction switched off, one should put the fine-structure constant equal to zero in the running of C_2 as well. This will lead to the running coupling $\tilde{C}_2(\mu)$ which cannot be calculated from $C_2(\mu)$. In fact the running of all couplings of low-energy effective field theory of strong and electromagnetic interactions depends on α. This dependence has to be switched off together with the explicit α-dependence if one considers the quantities of the theory with electromagnetic interaction switched off.

The underlying "fundamental theory" of strong interactions, QCD, is most likely an effective theory itself [20]. The only parameters of this theory which we can meaningfully interpret are the renormalized, running parameters. Therefore, the electromagnetic- and strong-interaction contributions in physical quantities cannot be unambiguously separated in this theory either (for detailed analyses see ref. [9]). The unambiguous separation of the electromagnetic- and strong-interaction contributions in physical quantities would be possible in a truly fundamental, non-perturbatively finite theory (string theory, M-theory?). On the other hand, if we consider the renormalized parameters of both QCD and QCD + electromagnetic interaction as an input, one can calculate (at least in principle) the low-energy constants of effective field theories with and without electromagnetic interaction using some non-perturbative technique like lattice calculations.

In conclusion, we have considered $^1S_0$ proton-proton scattering at very low energies in the framework of effective field theory, where all degrees of freedom except the proton and the photon are integrated out. We have argued that the dependence of the strong proton-proton scattering length (with the Coulomb interaction switched off) on the renormalization mass parameter occurs only if the Coulomb interaction is not completely switched off. To consider quantities entirely due to the strong interactions, one should also turn off the Coulomb interaction in the running of the strong-interaction coupling. Doing so generates the strong-interaction proton-proton scattering length which does not depend on the renormalization mass parameter.

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Appendix A.

In this appendix we illustrate the solution of the μ-dependence problem by means of a simple toy model. Our model is analogous to pp scattering in the sense that a "physical quantity" exhibits a renormalization scale dependence when one of the "coupling constants" is put equal to zero but is not simultaneously switched off in the running of the second coupling constant.

Suppose we have some "physical quantities"

$$y_1(p) = C_0 \ [a(p) + \alpha \ b(p)] , \tag{A.1}$$
$$y_2(p) = \alpha \ d_1(p) + C_0 \ d_2(p), \tag{A.2}$$

where α and C_0 are "coupling constants" and $a(p)$, $b(p)$, $d_1(p)$ and $d_2(p)$ are some given functions of "momentum" p. Let us express $y_2(p)$ in terms of α and a "renormalized coupling constant" $C_0(\mu) \equiv y_1(p)|_{p=\mu}$. Expressing C_0 as

$$C_0 = \frac{C_0(\mu)}{a(\mu) + \alpha \ b(\mu)} \tag{A.3}$$

and substituting eq. (A.3) into eq. (A.2), we obtain

$$y_2(p) = \alpha \ d_1(p) + \frac{d_2(p) \ C_0(\mu)}{a(\mu) + \alpha \ b(\mu)}. \tag{A.4}$$

Now, let $\tilde{y}_2(p)$ denote the result of $y_2(p)$ for $\alpha = 0$. Clearly, $\tilde{y}_2(p)$ determined from eq. (A.2) as $y_2(p)$ for $\alpha = 0$ does *not* depend on μ. On the other hand, if we naively substitute $\alpha = 0$ into eq. (A.4), we obtain

$$\tilde{y}_2(p) = \frac{d_2(p) \ C_0(\mu)}{a(\mu)}. \tag{A.5}$$

As a consequence, $\tilde{y}_2(p)$ determined from eq. (A.5) depends on μ, because the μ-dependence of $C_0(\mu)$ is *not* cancelled by μ-dependence of $a(\mu)$:

$$\frac{C_0(\mu)}{a(\mu)} = \frac{C_0 \left\{a(\mu) + \alpha b(\mu)\right\}}{a(\mu)}. \tag{A.6}$$

For this simple toy example the resolution of the seeming puzzle is clear: defining $\tilde{y}_2(p)$ in terms of "renormalized running coupling", we should substitute $\alpha = 0$ in eq. (A.4) and also replace $C_0(\mu)$ by $\tilde{C}_0(\mu)$, where $\tilde{C}_0(\mu) = y_1(\mu)$ for $\alpha = 0$. Doing so we obtain for $\tilde{y}_2(p)$

$$\tilde{y}_2(p) = \frac{d_2(p)\tilde{C}_0(\mu)}{a(\mu)}. \tag{A.7}$$

As $\tilde{C}_0(\mu) = a(\mu) \ C_0$, eq. (A.7) indeed gives $\tilde{y}_2(p)$ which (correctly) does not depend on μ.

The problem of the μ-dependence of the pp scattering length is fixed in analogy to this toy model. Note that $\tilde{C}_0(\mu)$ is uniquely determined by the "fundamental theory" and can be calculated in this toy model. In EFT $\tilde{C}_0(\mu)$ is again uniquely determined by the underlying theory but in practice it is not possible to calculate it (at least for the moment being) and therefore has to be given as an input.

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